On Computational Complexity of Vortex Element Method for 2D Incompressible Flows Simulation

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FSI-problems

FSI coupled problems

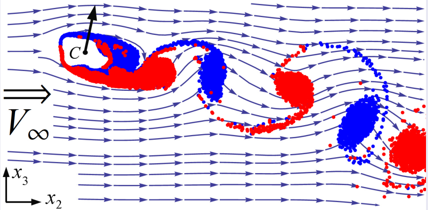
- Movable bodies
- Deformable bodies

} Variable flow region }

- How to construct mesh?
- How to satisfy BC?

Main assumptions

- 2D flows are considered
- Flow is viscous incompressible
- Airfoils are heavy ($\rho_0/\rho \gg 1$)
  ρ_0 — airfoil's average density
  ρ — density of the flow



On the Development of Vortex Methods

Scientific schools

- S.M. Belotserkovskii, M.I. Nisht, I.K. Lifanov, A.V. Setukha (Zhukovsky Air Force Engineering Academy, MSU);
- S.V. Guvernyuk, G.Ya. Dynnikova (MSU);
- V.I. Morozov, V.A. Aparinov (Scientific and research inst. of parachute design and prod.);
- M.A. Golovkin, V.M. Kalyavkin (Central Aerohydrodynamic Institute);
- G.A. Shcheglov, I.K. Marchevsky (BMSTU);
- D.N. Gorelov (Siberian Branch of the Russian Academy of Sciences, Tomsk);
- S.A. Dovgiy, D.I. Cherniy (National Academy of Sciences of Ukraine, Kiev);
- A. Leonard, G. Winckelmans (Belgium);
- G.-H. Cottet, P. Koumoutsakos (France, Switzerland);
- G. Morgenthal (Great Britain).

Conferences on Vortex Methods

- International conference on vortex flows and models (2010, 2016);
- International conference ‘Coupled Problems’ (ECCOMAS) (2013, 2015);
- International symposium ‘Method of discrete singularities in math. physics’ (2003 – 2013);
- International seminar named after S.M. Belotserkovskii (2004 – 2015);

Features of Vortex Method

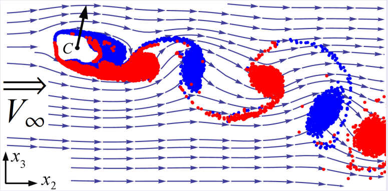
- Lagrangian meshless method (Particle method);
- easy to solve coupled FSI problems;
- small numerical viscosity;
- possibility to obtain acceptable results for practical problems with small computational burden;
- application area of vortex method:
  - airframe and parachute engineering;
  - aircraft vortex wake calculation;
  - modeling of the main and tail helicopter rotors;
  - industrial aerodynamics of buildings, urban aeration.

Disadvantages of vortex method

- application area: incompressible flow;
- law accuracy of vortex sheet computation;
- need to consider pair influences for all vortex elements (as in N-body problem).

Equations of fluid motion and boundary conditions

Continuity & Navier — Stokes equations:



$$\nabla \cdot \vec{V} = 0, \qquad \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = \nu \Delta \vec{V} - \frac{\nabla p}{\rho}.$$

Boundary conditions:

Conditions of perturbations decay at infinity:

$$\vec{V}(\vec{r}, t) \to \vec{V}_\infty, \quad p(\vec{r}, t) \to p_\infty, \quad |\vec{r}\,| \to \infty.$$

No-slip condition: $\vec{V}(\vec{r}, t) = \vec{V}_K(\vec{r}, t), \quad \vec{r} \in K.$

$\vec{V}$ — velocity in the flow;
p — pressure;
$\vec{V}_\infty$, p_∞ — velocity and pressure at infinity;
ν — kinematic viscosity coefficient.

Basic Ideas

Viscous Vortex Domains (VVD) method

- Vorticity $\vec{\Omega}(\vec{r}, t) = \nabla \times \vec{V}(\vec{r}, t)$ — primary compute variable.
- Navier — Stokes equations in Helmholtz form:

$$\frac{\partial \vec{\Omega}}{\partial t} + \nabla \times (\vec{\Omega} \times (\vec{V} + \vec{W})) = 0.$$

- It can be treated as transport equation for $\vec{\Omega}$, which moves in velocity field $\vec{V} + \vec{W}$,

$$\vec{W}(\vec{r}, t) = -\nu \frac{\nabla \Omega}{\Omega} \text{ — diffusive velocity}, \quad \Omega = \vec{\Omega} \cdot \vec{k}.$$

- No vorticity generation in flow region.
- New vorticity is generated only on the camber line of the airfoil.

1. Dynnikova, G.Ya. The Lagrangian approach to solving the time-dependent Navier — Stokes equations. *Doklady Physics.* (2004) **49**: 648–652.

Airfoil Influence

Vortex and source Sheets

Airfoil influence is simulated by attached vortex and source sheets and free vortex sheet placed on its camber line:

- Intensity of the attached vortex sheet

$$\gamma_{\text{att}}(\vec{r},\, t) = \vec{V}_K(\vec{r},\, t) \cdot \vec{\tau}(\vec{r},\, t), \quad \vec{r} \in K.$$

- Intensity of the attached source sheet

$$q_{\text{att}}(\vec{r},\, t) = \vec{V}_K(\vec{r},\, t) \cdot \vec{n}(\vec{r},\, t), \quad \vec{r} \in K.$$

- Intensity of the free vortex sheet $\gamma(\vec{r},\, t)$ can be determined from the boundary condition satisfaction.

$\vec{\tau}(\vec{r},\, t)$ и $\vec{n}(\vec{r},\, t)$ — unit normal and tangent vectors.

Flow Velocity

Generalized Biot — Savart law

$$\vec{V}(\vec{r},t) = \vec{V}_\infty + \frac{1}{2\pi}\int_{S(t)} \frac{\vec{\Omega}(\vec{\xi},t)\times(\vec{r}-\vec{\xi})}{|\vec{r}-\vec{\xi}|^2}dS + \frac{1}{2\pi}\oint_{K(t)} \frac{\vec{\gamma}(\vec{\xi},t)\times(\vec{r}-\vec{\xi})}{|\vec{r}-\vec{\xi}|^2}dl_K +$$

$$+\frac{1}{2\pi}\oint_{K(t)} \frac{\vec{\gamma}_{\text{att}}(\vec{\xi},t)\times(\vec{r}-\vec{\xi})}{|\vec{r}-\vec{\xi}|^2}dl_K + \frac{1}{2\pi}\oint_{K(t)} \frac{q_{\text{att}}(\vec{\xi},t)(\vec{r}-\vec{\xi})}{|\vec{r}-\vec{\xi}|^2}dl_K,$$

$$\vec{\gamma}_{\text{att}} = \gamma_{\text{att}}\vec{k}, \qquad \vec{\gamma} = \gamma\vec{k}, \qquad \vec{\Omega} = \Omega\vec{k}, \qquad \vec{n}(\vec{r},\, t)\times\vec{\tau}(\vec{r},\, t) = \vec{k}.$$

Limit value of the velocity on the airfoil camber line

$$\vec{V}_-(\vec{r},\, t) = \vec{V}(\vec{r},\, t) - \frac{\gamma(\vec{r},\, t)-\gamma_{\text{att}}(\vec{r},\, t)}{2}\vec{\tau}(\vec{r},\, t) + \frac{q_{\text{att}}(\vec{r},\, t)}{2}\vec{n}(\vec{r},\, t)$$

1. Zhukovsky N.E. On attached vortices. 1908.
3. Kempka S.N., Glass M.W., Peery J.S., Strickland J.H. Accuracy considerations for implementing velocity boundary conditions in vorticity formulations // SANDIA REPORT. SAND96-0583, UC-700, 1996. 52 p.

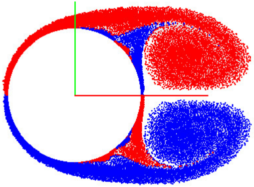
Numerical Approximation

Vortex wake simulation

Vorticity distribution in the flow is simulated by large number of separate vortex elements (VE)

$$\Omega(\vec{r}) = \sum_{i=1}^{n} \Gamma_i \delta(\vec{r} - \vec{r}_i),$$

Γ_i — circulation of the VEs, $\vec{r}_i$ — their positions.



Vortex elements movement

Movement equation: $\dfrac{D\vec{\Omega}}{Dt} = 0 \quad \Leftrightarrow \quad \begin{cases} \Gamma_i = \mathrm{const}, \\ \dfrac{d\vec{r}_i}{dt} = \vec{V}(\vec{r}_i) + \vec{W}(\vec{r}_i), \quad i = 1, \dots, n. \end{cases}$

$$\vec{V}(\vec{r}_i) = \sum_{\substack{j=1 \\ j \neq i}}^{n} \underbrace{\frac{\Gamma_j}{2\pi} \frac{\vec{k} \times (\vec{r}_i - \vec{r}_j)}{|\vec{r}_i - \vec{r}_j|^2}}_{\vec{v}_{ij}} + \vec{V}_\gamma + \vec{V}_\gamma^{\mathrm{att}} + \vec{V}_q^{\mathrm{att}} + \vec{V}_\infty.$$

Free Vortex Sheet Intensity Computation

No-slip boundary condition

$$\vec{V}_-(\vec{r},t) = \vec{V}_K(\vec{r},t) \quad \Leftrightarrow \quad \underbrace{\vec{V}\cdot\vec{n} = \vec{V}_K\cdot\vec{n}}_{N\text{-scheme}} \quad \Leftrightarrow \quad \underbrace{\vec{V}_-\cdot\vec{\tau} = \vec{V}_K\cdot\vec{\tau}}_{T\text{-scheme}}$$

N-scheme: Singular integral equation

T-scheme: Fredholm (2^{nd} kind) integral equation

N-scheme Lifanov, I.K., Belotserkovskii, S.M. *Methods of Discrete Vortices*. CRC Press, 1993.

T-scheme Kempka, S.N., Glass, M.W., Peery, J.S. and Strickland, J.H. Accuracy Considerations for Implementing Velocity Boundary Conditions in Vorticity Formulations. *SANDIA Report SAND96-0583*, 1996.

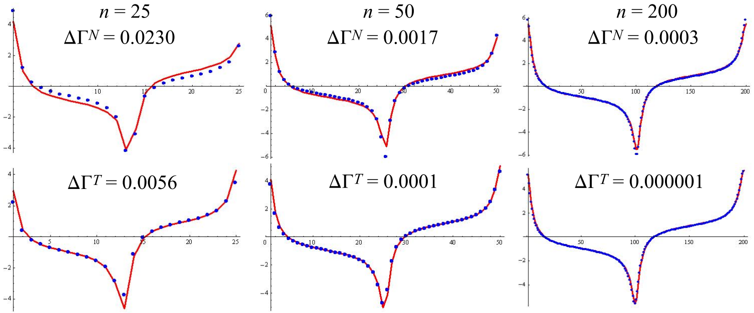
Cottet G.-H., Koumoutsakos P.D. Vortex Methods: Theory and Practice. — Cambridge University Press, 2008. — 328 p.

T-scheme allows to obtain much more accurate results when solving FSI-problems

1. Marchevsky, I.K. and Moreva, V.S. Vortex Element Method for 2D Flow Simulation with Tangent Velocity Components on Airfoil Surface. *ECCOMAS 2012 — European Congr. on Comp. Meth. in Appl. Sc. and Eng., e-Book.* (2012) 5952–5965.
2. Kuzmina K.S., Marchevsky I.K. The Modified Numerical Scheme for 2D Flow-Structure Interaction Simulation Using Meshless Vortex Element Method // PARTICLES 2015: Proc. of the IV Inter. Conf. on Particle-Based Methods (Barcelona, Spain, 28 – 30 Sept. 2015). — 2015. Pp. 680–691.

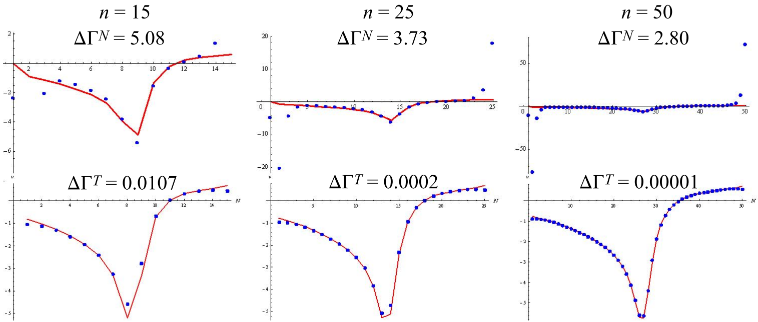
Comparison of N- and T-scheme accuracy for flow simulation around elliptical airfoil

- $a = 1{,}0$, $b = 0{,}1$ — ellipse semiaxes.
- $\alpha = \pi/6$ — incident flow angle.



Comparison of N- and T-scheme accuracy for flow simulation around Zhukovsky airfoil

- $a = 3{,}5$, $d = 0{,}4$, $h = 0{,}3$ — Zhukovsky airfoil parameters.
- $\alpha = \pi/6$ — incident flow angle.



Goals and Objectives

Goal

The estimation of computational complexity of the vortex method and its effective implementation.

Objectives

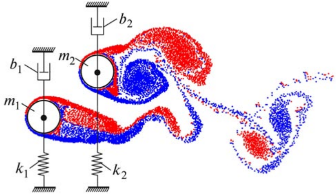
- 'Typical' model problems statement.
- Analysis of the computational complexity of the main algorithm operations.
- Analysis of the possible ways to accelerate calculations.
- Optimized algorithm complexity estimation.
- Model problems solving.

Model problems description. Problem 1

Simulation of hydroelastic oscillations of two cylinders

'Base' parameters of numerical scheme:

- $\mathbf{n_{p0}} = 200$ — number of vortex elements that simulate vortex sheet on one cylinder;
  $\mathbf{n_0} = 2 \cdot n_{p0} = 400$ — total number of vortex elements that simulate vortex sheet.
- $\mathbf{N_{p0}} = 10\,000$ — number of vortex elements that simulate vortex wake near one cylinder;
  $\mathbf{N_0} = 20\,000$ total number of vortex elements that simulate vortex wake.
- $\mathbf{T_0} = 30\,000$ — number of required time steps.



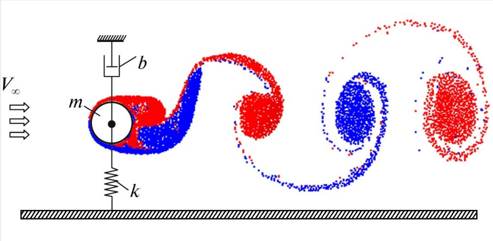
For arbitrary n_p:

$$n = 2n_p, \; N = 2N_0\left(\frac{n_p}{n_{p0}}\right)^2, \; T = T_0 \cdot \left(\frac{n_p}{n_{p0}}\right).$$

Model problems description. Problem 2

Simulation of hydroelastic oscillations of cylinder in presence of shielding surface

Vortex layer on the shielding surface remains to be attached.



'Base' parameters of numerical scheme:

$\mathbf{n_{p0}} = 200$ VE — vortex sheet on the cylinder;

$\mathbf{n_{e0}} = 3n_{p0} = 600$ VE — vortex sheet on the shielding surface;

$\mathbf{N_0} = 10\,000$ VE — vortex wake.

Parameters for arbitrary n_p

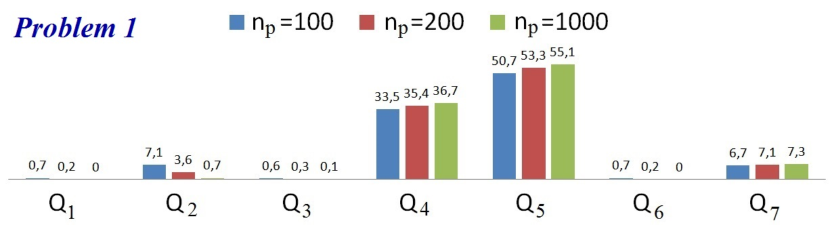
$$n = 4n_p, \quad N = N_0 \cdot \left(\frac{n_p}{n_{po}}\right)^2, \quad T = T_0 \cdot \left(\frac{n_p}{n_{p0}}\right).$$

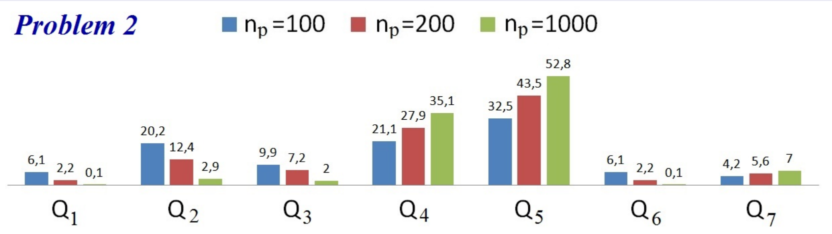
Computational complexity for main operations

- **Operation 1.** SLAE matrix formation for generated vortex elements circulations determination.
$$Q_1^N = 6n^2, \quad Q_1^T = 83n^2.$$
- **Operation 2.** SLAE right hand side calculation.
$$Q_2^N = 7Nn + 10n^2, \quad Q_2^T = 30Nn + 85n^2.$$
- **Operation 3.** SLAE solving or multiplying by inverse matrix.
$$Q_3 = n^3/3 \quad \text{or} \quad Q_3 = n^2.$$
- **Operation 4.** Calculation of convective velocities of vortex elements.
$$Q_4 = 6N^2 + 8Nn.$$
- **Operation 5.** Calculation of diffusive velocities of vortex elements.
$$Q_5 = 9N^2 + 14Nn.$$
- **Operation 6.** No-through control block.
$$Q_6 \approx Q_1.$$
- **Operation 7.** Vortex wake reconstruction.
$$Q_7 \approx 0{,}2Q_4.$$

$$Q = \frac{n^3}{3} + 251\,n^2 + 53{,}6\,Nn + 16{,}2\,N^2.$$

Operations $Q_1 \ldots Q_7$ (%)





Computational complexity of model problems

Total numerical complexity of model problems

$$S_1(200) = Q_1(200) \cdot T_0 \approx 2,1 \cdot 10^{14},$$

$$S_2(200) = Q_2(200) \cdot T_0 \approx 7,1 \cdot 10^{13}$$

.

n_p	100	200	400	600	800	1 000
$\dfrac{S_1(n_p)}{S_1(200)}$	0,03	1	31	231	969	2946
$\dfrac{S_2(n_p)}{S_2(200)}$	0,05	1	26	188	766	2293

Ways to accelerate computations

- Use parallel computing.
- Use approximate fast methods:
  - fast multipole method;
  - mosaic-skeleton approximations (E.E. Tyrtyshnikov);
  - auxiliary Poisson equation solving.

Usage of parallel computing technologies for vortex method implementation

'Old' implementation

In [*] algorithm with parallel implementation of operations 4, 5, 6, 7 was described.

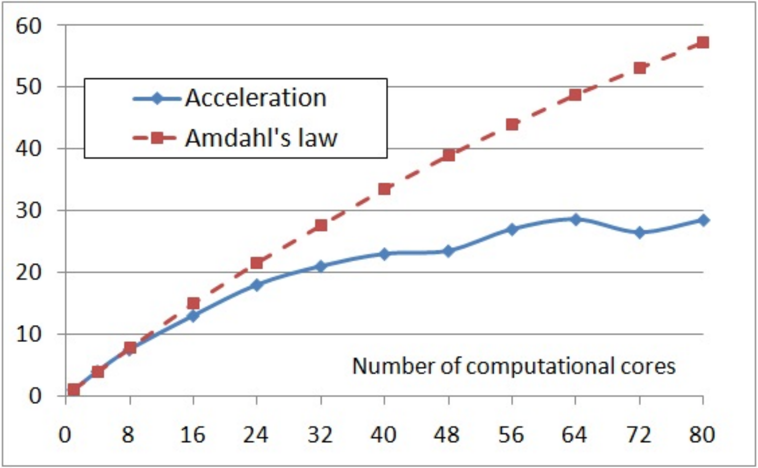
1 problem. Speeding up 25-30 times in the calculations on the 64-core cluster. Amdahl law predicts 45 times acceleration.

2 problem. Speeding up 5 times in the calculations on the 64-core cluster. Amdahl law predicts 7.5 times acceleration.

Parallel implementation of 4 operations is not enough! It is necessary to develop parallel algorithms for all seven major operations of vortex method!

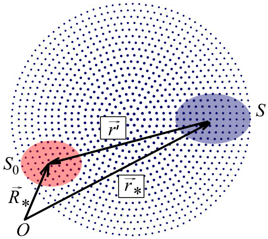
[*] Марчевский И.К., Морева В.С. Параллельный программный комплекс POLARA для моделирования обтекания профилей и исследования расчетных схем метода вихревых элементов // Параллельные вычислительные технологии (ПаВТ'2012): Труды международной научной конференции (Новосибирск, 26–30 марта 2012 г.). Челябинск: Издательский центр ЮУрГУ, 2012. С. 236–247.

Acceleration with 'old' parallel implementation

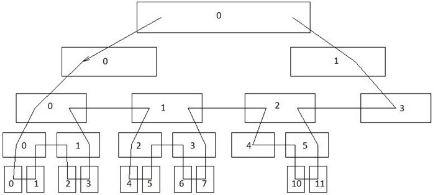


Fast method (by analogy with N-body problem)

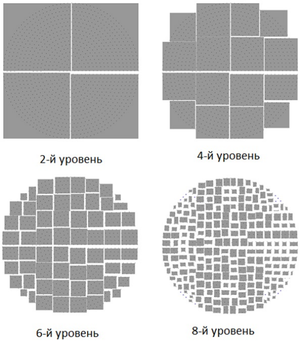
Barnes J., Hut P. A hierarchical $O(N \log N)$ force-calculation algorithm // Nature. 1986. V. 324, No. 4. P. 446-449.



- Collective influence of remote VE is calculated approximately.
- The basis of the algorithm — creation of the hierarchical tree structure.



Tree structure and traversal scheme



Velocities of all vortex elements in considered cell of lower level

$$\vec{V}_i \approx \begin{pmatrix} A \\ B \end{pmatrix} + \begin{pmatrix} C & D \\ D & -C \end{pmatrix} \begin{pmatrix} \Delta x_i \\ \Delta y_i \end{pmatrix} + \sum_j \vec{v}_{ij} + \vec{V}_\infty$$

A, B, C и D — calculated coefficients, which are common for all VE in cell.

Algorithm parameters

- θ — cells proximity parameter ($0 \leqslant \theta \leqslant 4$), determined by required accuracy. Proximity condition: $|\vec{r}'| \le \dfrac{h + h_0}{\theta}$.
- k — maximal tree level.

Computational complexity

Direct calculation	Fast method
$O(N^2)$	$O(N \log N)$ with optimal k

Approximate estimation of computational complexity of fast algorithm

Number of multiplications and divisions in convective velocities computation using fast method (Q_4)

$$Q_4^{fast} = \frac{24N^2}{2^k}\left(\frac{4}{\theta}\right)^2\left(1-\alpha\frac{(\sqrt{2})^k-1}{\sqrt{N}}\right)^2\left(1-\frac{4}{\theta(\sqrt{2})^k}\left(1-\alpha\frac{(\sqrt{2})^k-1}{\sqrt{N}}\right)\right)+$$

$$+\frac{896\cdot 2^k\cdot\beta}{\theta^2}\left(4\left(\frac{1}{4+\theta}+\frac{1}{4-(\sqrt{2})^k\theta}\right)+\ln\left(\frac{(\sqrt{2})^k-4}{4+\theta}\right)\right)+4N.$$

N — number of VE in flow region,
α, β — experimentally selected parameters ($\alpha = 0.84$, $\beta = 0.56$).
$\theta \approx 0,4$ gives $0,1\ldots,0,2$ % error.

Кузьмина К.С., Марчевский И.К. Оценка трудоемкости быстрого метода расчета вихревого влияния в методе вихревых элементов // Наука и образование: электронное научно-техническое издание. 2013. № 10. С. 399–414.

Diffusive velocities computation using fast method (Q_5)

$$Q_5 = Q_4\big|_{\theta=\theta_{dif}} \cdot N \cdot \theta_{dif} \cdot \frac{\gamma}{2^k}.$$

$\gamma = 0,7$ — empirical coefficient; θ_{dif} — parameter which determines the accuracy of method, k — number of layers.

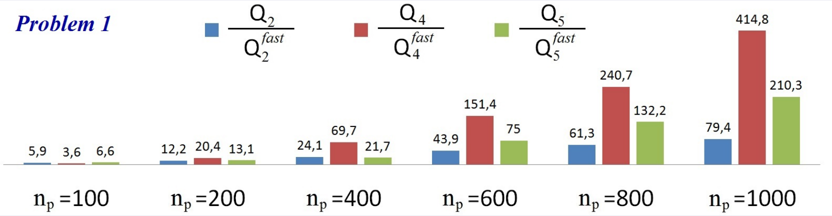
$\theta_{dif} \approx 0,1$ gives $0,1\ldots,0,2$ % error.

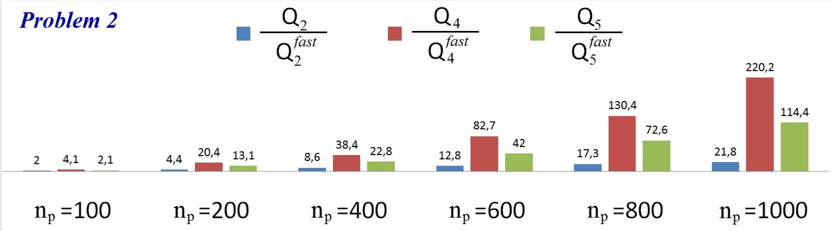
SLAE right-hand side calculation (Q_2)

$$Q_2^{fast} = \frac{130Nn}{2^k}\left(\frac{4}{\theta}\right)^2\left(1-\alpha\frac{(\sqrt{2})^k-1}{\sqrt{N}}\right)^2\left(1-\frac{4}{\theta(\sqrt{2})^k}\left(1-\alpha\frac{(\sqrt{2})^k-1}{\sqrt{N}}\right)\right)+$$
$$+\frac{896\cdot n\cdot\beta}{\theta^2}\left(4\left(\frac{1}{4+\theta}+\frac{1}{4-(\sqrt{2})^k\theta}\right)+\ln\left(\frac{(\sqrt{2})^k-4}{4+\theta}\right)\right)+85n^2.$$

Кузьмина К.С., Марчевский И.К. Об оценках вычислительной сложности и погрешности быстрого алгоритма в методе вихревых элементов // Труды Института системного программирования РАН. 2016. Т. 28. № 1. В печати.

Acceleration of operations Q_2, Q_4, Q_5 using fast method





Other operations of algorithm

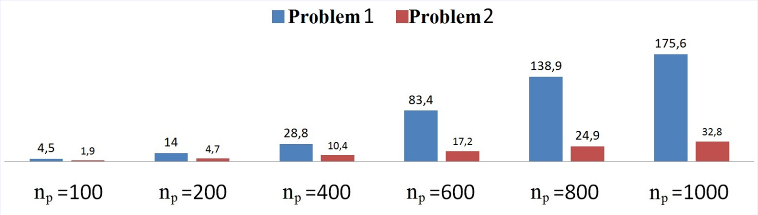
Operations 6 (no-through control) and 7 (vortex wake reconstruction) also can be accelerated using tree structure:

$$Q_6^{fast} = Q_1^{fast}, \qquad Q_7^{fast} = 0{,}2\,Q_4^{fast}.$$

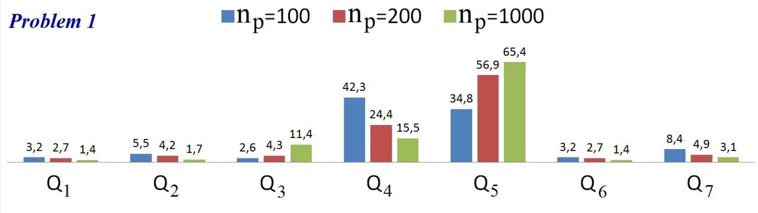
Operations 1 (SLAE matrix formation) and 3 (SLAE solving) cannot be accelerated using fast method:

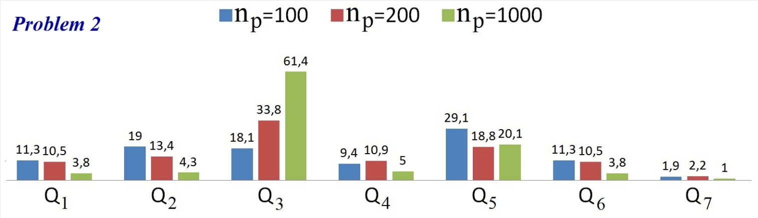
$$Q_1^{fast} = Q_1, \qquad Q_3^{fast} = Q_3.$$

Computational complexity S (direct calculation) with respect to S^{fast} (using fast method)



Operations $Q_1 \dots Q_7$ (%) when using fast method



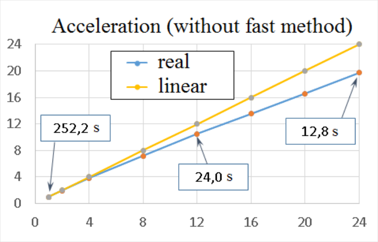


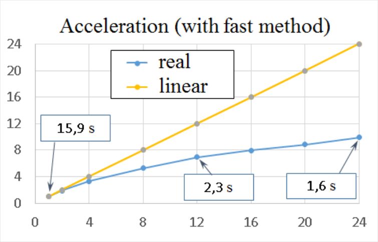
Parallel implementation of the fast algorithm

'New' algorithm implementation

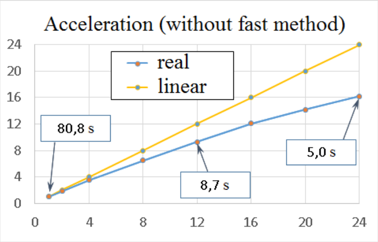
- **1, 2 and 6 operations** parallelized using MPI technology.
- **3 operation** implemented using Eigen library (linear algebra library) using OpenMP.
- For **4 and 5 operations** fast algorithms are implemented, which parallelized using MPI.
- For **7 operation** effective algorithm are implemented using tree structure and MPI technology.

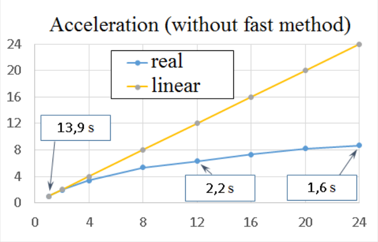
Problem 1





Problem 2





Conclusions

- It is shown that computational complexity distribution over the operations depend heavily on problem statement.
- The computational complexity estimations of the fast algorithm are obtained; these estimations allow to choose optimal parameters for algorithm.
- Parallel implementation of vortex method using fast method is developed.